The Whole Story on the Hole Effect

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The spatial continuity of the variables we model in geostatistics is dependent on the modelling and reproduction of the variogram. The variogram defines the relationship between variability (or geologic distance) and the lag distance (or Euclidian distance).

As the magnitude of the lag separation vector increases, we typically expect the variogram to also increase. This is generally observed. The majority of variogram model structures are monotonic increasing. Nevertheless, non-monotonic structures may have a physical interpretation, provide valuable information and may be modeled with positive definitive models for more accurate geologic models. Non- monotonic variograms that show cyclic patterns are called "hole effect" variograms.

The theory and application of hole effect structures is explored in this report. Efforts are made to illustrate and provide examples of the hole effect structure. The generation of simulated realizations in presence of hole effect structures is investigated, with special attention paid to the reproduction of the histogram and variogram and their ergodic fluctuations. Hole effect variograms are commonly encountered and may be used in geostatistical simulation. The variogram structures are reproduced together with the histogram and the local conditioning data.

KEYWORDS: geostatistical simulation, stochastic modeling, variogram modeling

The Origin

Experimental variograms often continuously increase with lag distance; however, the variogram is not restricted to such monotonic form and decreasing segments or cyclicity can be observed. Figure!1 shows two examples to illustrate cyclic variograms. The first data set is a sequence of regular lenses, while the second data set is a set of regular horizontal beds. These configurations result in cyclic features in the experimental variograms (see Figure 1).

Non-monotonic variogram structures are identified as "hole effect" structures (Journel and Huijbregts, 1978). These structures may be bounded by a sill or occur without a sill, be dampened or undampened and be isotropic or anisotropic (see Figure 2). Although hole effect structures are often ignored, their presence provides valuable information concerning spatial variability. Hole effect structures most often indicate a form of cyclicity or periodicity, which is a common and legitimate spatial characteristic in geology. Ignoring these non-monotonic structures may result in unrealistic heterogeneity models that do not reproduce the observed patterns of varibility.

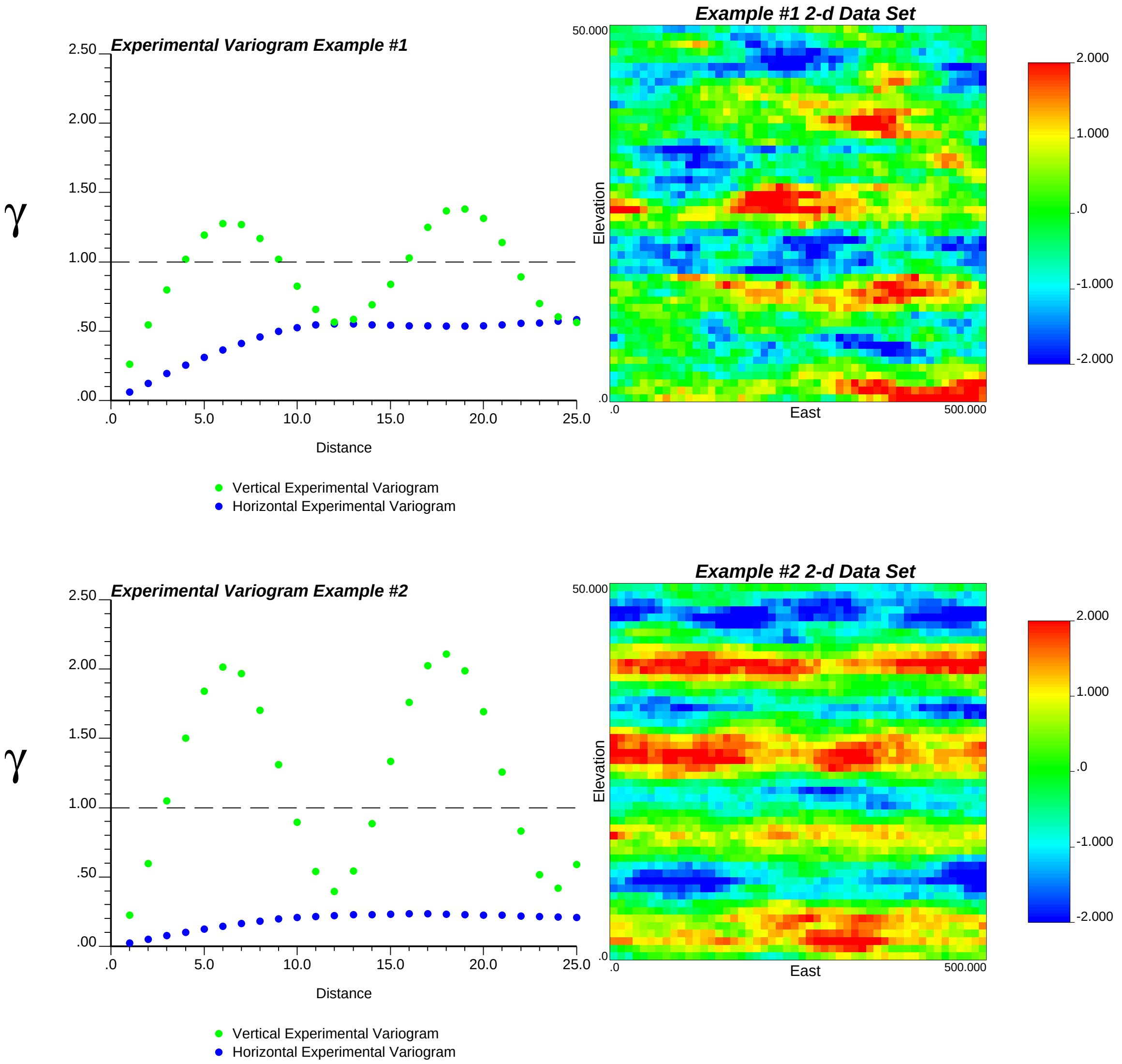

Figure 1: Two example data sets with non-monotonic continuity structures

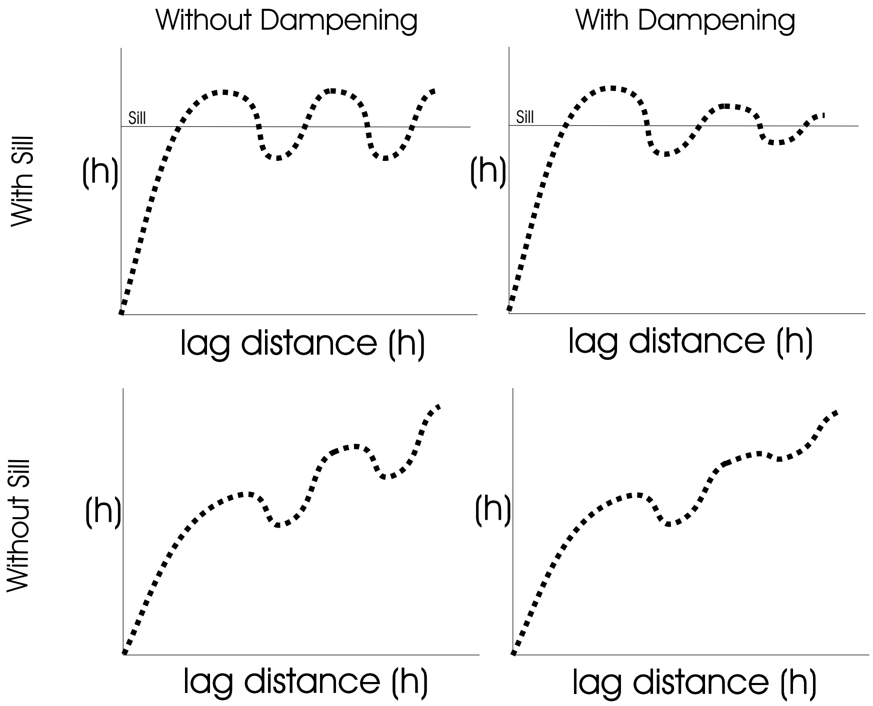

Figure 2: Hole effect configurations: The observed cyclicity may be dampened. In this case the amplitude is reduced with each subsequent wavelength. The variogram may be bounded by a sill occur without a sill.

Interpretations

The form and character of hole effect structures is indicative of the spatial setting. After observing many spatial configurations and their respective variograms, the following generalizations are reached (see Appendix A for the data sets and variograms):

Regular clustered lenses: (1) the lag distance at the first peak is an indication of the average extent of the lenses in the specific lag direction, (2) the lag distance at the first trough is the sum of the average extent of the lenses and the average distance between lenses in the specific lag direction, and (3) the distance from the peak to the trough is the average lens spacing.

Regular strata: (1) the lag distance of the first peak is an indication of the average thickness of the bedding, and (2) the lag distance of the first trough is twice the "bed" thickness.

Nonclustered (randomly located) lenses: (1) the lag distance at the first peak (or the sill if no peak is visible)is an indication of the average extent of the lenses in that direction, and (2) if the data set is large relative to the lens size, there is no cyclicity observed on the variogram.

Clustered irregular lenses: similar to clustered regular lenses but the variation in lens size and lens spacing results in the peaks being attenuated and the cyclicity being dampened.

Irregular strata: (1) cyclicity is observed if there is a continuously repeating series and the peaks are attenuated over the distribution of bedding widths in each unit sequence, and (2) in the absence of a perfectly repeating series, dampening occurs.

These observations are illustrated in Figures 3 and 4 and in the examples provided in Appendix A.

In the presence of cyclicity the variogram exceeds the sill. The sill represents the global variance. The link between the variogram and covariance relationship helps our interpretation of the variogram above the sill.

$$\begin{aligned}\gamma(\mathbf{h}) &= C(0) - C(\mathbf{h}) \\ &= C(0) \cdot [1 - \rho(\mathbf{h})]\end{aligned}$$

where $\gamma(\mathbf{h})$ is greater than $C(0)$ or the sill for any $rho(\mathbf{h}) \in [-1, 0)$. When the variogram exceeds the sill the correlation is negative between locations separated by lag vector $\mathbf{h}$.

Analytical Hole Effect Variogram Models

To build accurate geostatistical models it is necessary that the significant observed hole effect structures be reproduced. This requires the construction of a legitimate variogram model that reflects the observed hole effect.

$$\gamma(h) = c_n \cdot [1.0 - cos(\frac{h}{a_n}\pi)]$$

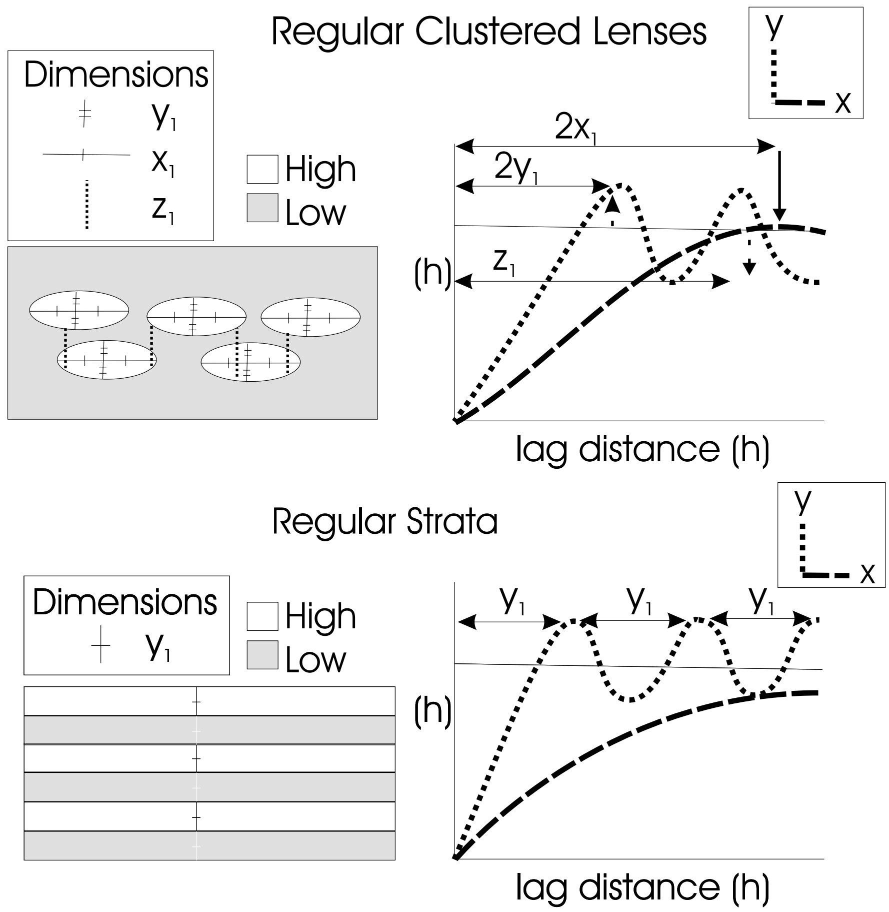

Figure 3: Hole effect interpretations in regular clustered lenses and regular strata based on indicator data

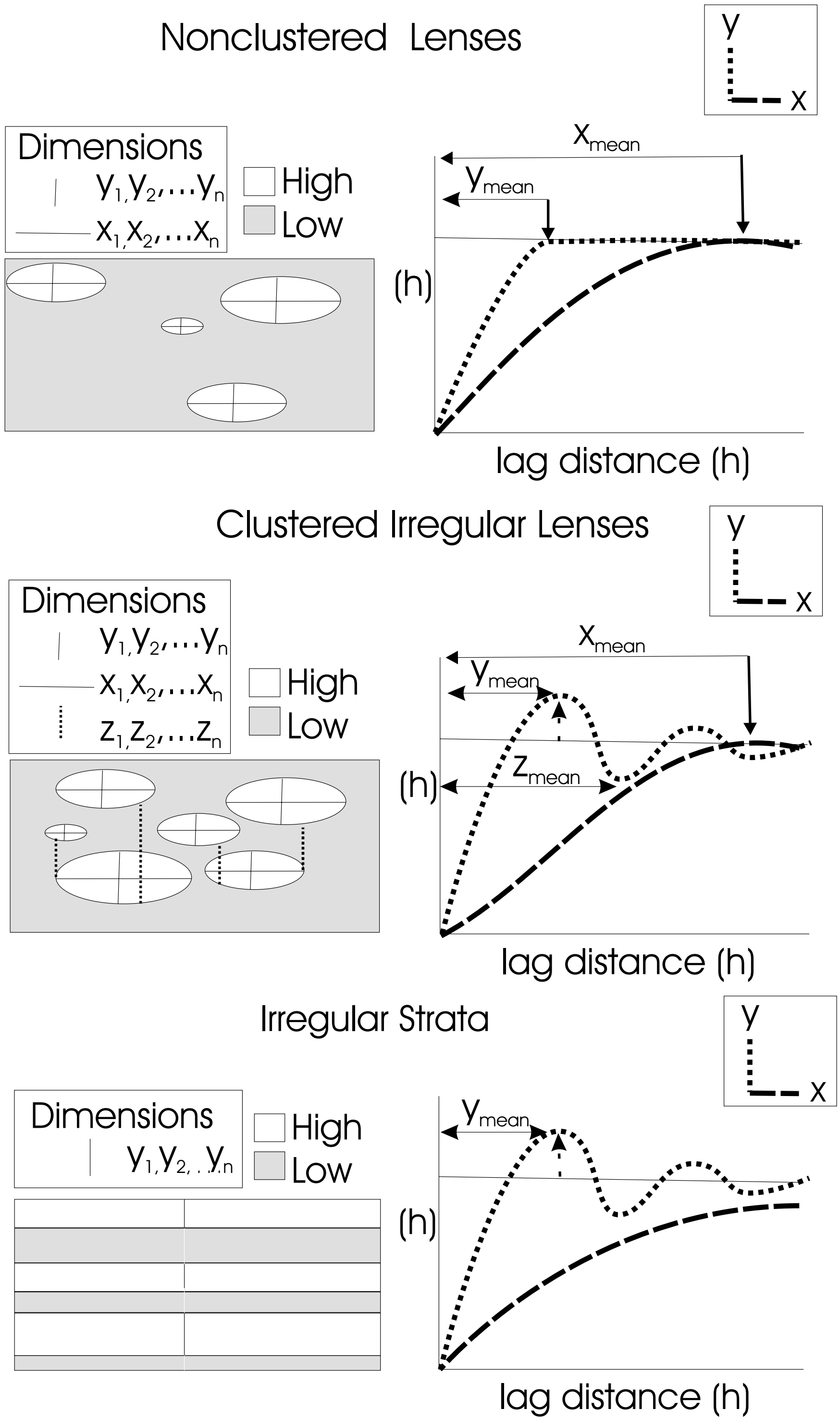

Figure 4: Hole effect interpretations in irregular lenses and irregular strata based on indicator data

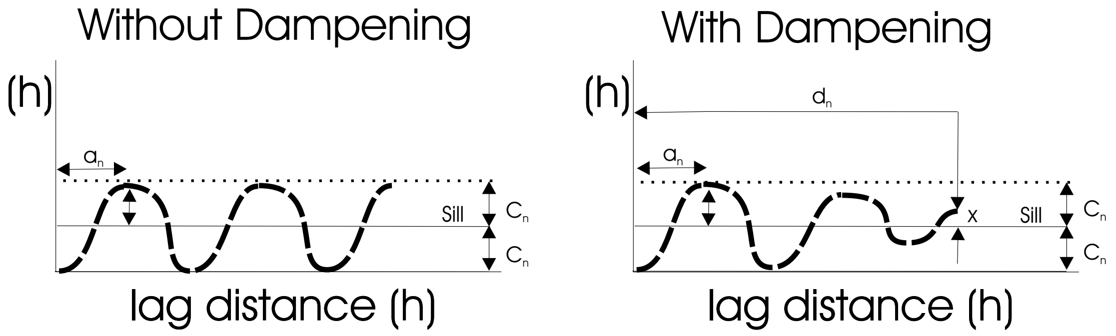

Figure 5: Parameters associated with the hole effect model

The parameters include the variance contribution, c_n, and the range, a_n, which is the distance to the first peak. A dampened model would also require a dampening parameter, d_n. This is generally set as the range at which the relative amplitude is equal to a specific fraction, x, of the original amplitude, $x \cdot \alpha$. It should be noted that currently the dampened model is present in the GSLIB code, but it is "commented out". For an illustration of these parameters, see Figure 5. The following are considerations with respect to derivation of a hole effect variogram model.

Positive Definiteness

The covariance counterpart to the variogram must be positive definite. Positive definite models ensure that variance is non-negative, and that the variogram is a legitimate measure of geologic distance. There are a variety of positive definite models available to model the hole effect. These models are positive definite in a specific spatial dimension.

The sine and cosine functions are natural choices for nested structures, which express cyclicity. In general the sine model $C(h) = sin(h)$ is positive definite in $3d$, while the cosine model $C(h) = cos(h)$ is only positive definite in $1d$ space. The sine model may be used in one or two dimensional space since positive definiteness in p-d space guarantees positive definiteness in n space (where $p \geq n$).

The cosine model must be restricted to one dimension. This can be accomplished by setting the range parameters (a_x, a_y, a_z) to very large values in all directions except for the direction of the hole effect structure (Deutsch and Journel, 1998). As shown below, by setting the range parameter a_x and a_y to very large numbers the nested structure, which is a function of h_x, h_y and h_z is reduced to a function of h_z:

$$h_{(x,y,z)} = \sqrt{\left(\frac{h_x}{a_x}\right)^2 + \left(\frac{h_y}{a_y}\right)^2 + \left(\frac{h_z}{a_z}\right)^2}$$

$$as\ a_x, a_y \overset{\infty}{\longrightarrow} h_{(x,y,z)} = \sqrt{\left(\frac{h_z}{a_z}\right)^2} = \frac{h_z}{a_z}$$

The Variogram Model Must Honor the Correct Sill

Stationary simulation algorithms rely on variograms which are modeled such that the sum of the contributions of the nested structures is equal to global variance. Most simulation algorithms are for stationary phenomena.

Isotropic and Anisotropic Models

Hole effect structures can occur with isotropic and anisotropic phenomenon. The isotropic model is limited in its application. It has been stated that the maximum relative amplitude in an isotropic hole effect model is 0.212 (Journel and Huijbregts, 1978); however, this if for the 3-D model, which is rarely used.

Without Dampening and Dampened Models

The hole effect is generally dampened. This is caused by irregularities in feature intervals and by the superposition of other continuity structures. Dampening is achieved by multiplying the covariance function by an exponential covaraince, that acts as a dampening function.

$$\gamma(h) = c \cdot [1.0 - exp^{(\frac{-3h}{d})} \cdot cos(\frac{h}{a}\pi)]$$

In variography, we take advantage of the fact that the positive sum of nested positive definite models results in a positive definite composite model. The fact that the product of positive definite covaraince models is positive definite is less utilized.

The GSLIB Model

The previously mentioned undampened cosine model is currently utilized in GSLIB. This model is supported as nested structure number 5 in all the component programs. A complete illustration of its behavior in variogram and covariance space is shown in Figure 6. The GLSIB hole effect is modeled with the previously mentioned parameters.

$$\gamma(h) = c_n \cdot [1.0 - cos(\frac{h}{a_n}\pi)]$$

The Dampened GSLIB Model

The "commented out" dampened GSLIB hole effect model takes the form of the previously mentioned exponential and cosine combination. The additional parameter, d, is the lag distance at which 95% of the hole effect oscillation is dampened. A complete illustration of its behavior in variogram and covariance space is shown in Figure 7.

$$\gamma(h) = c \cdot [1.0 - exp^{(\frac{-3h}{d})} \cdot cos(\frac{h}{a}\pi)]$$

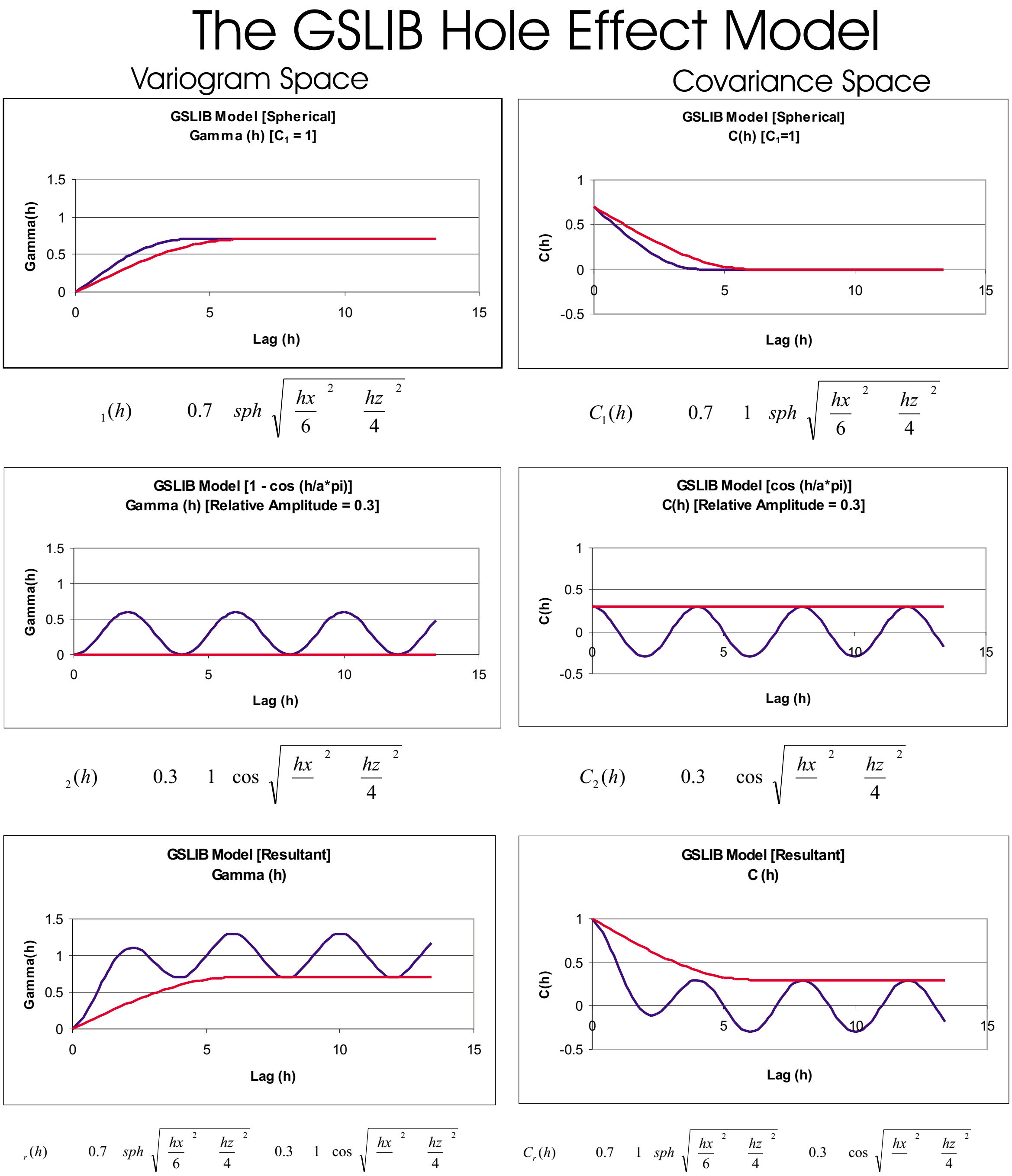

Figure 6: The GSLIB hole effect model

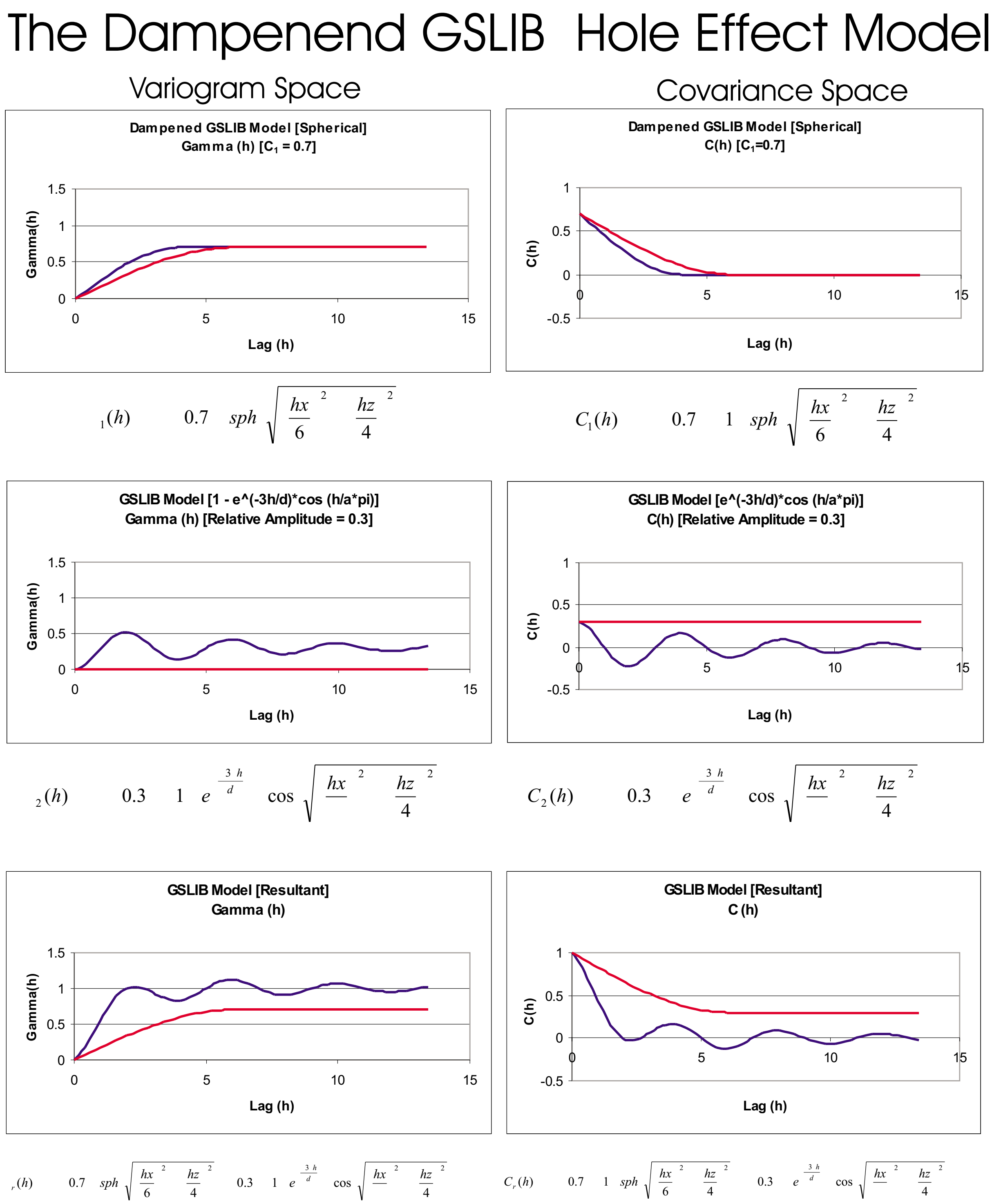

Figure 7: The dampened GSLIB hole effect model

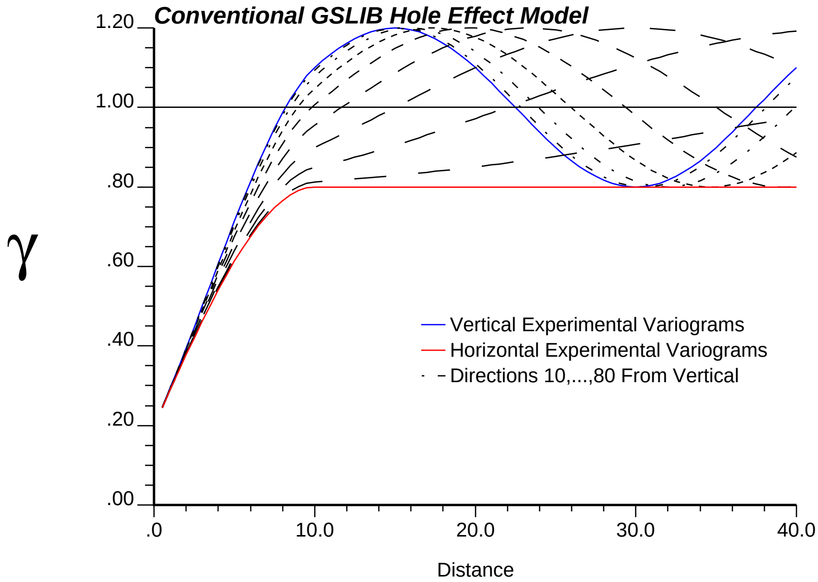

Figure 8: The conventional GSLIB hole effect model at angles $0, \ldots, 90$ degrees

The Limitations of Hole Effect Modelling

The variogram modeling, kriging and simulation components of GSLIB only recognize the undampened cosine hole effect model described above. This model is displayed in Figure 8 with angles $0, 10, \ldots 90$ plotted (although modification of the variogram models is possible by making straightforward changes to the FORTRAN code). The current GSLIB model has the following limitations:

- The hole effect nested structure may only exist in one direction. The nested structure is limited to 1-D space. This causes the variogram to be dependent on only one component of the 3-D lag vector, **h**. This limitation is not significant, since most empirical hole effect structures are only observed in one direction.
- The absence of dampening is a significant limitation. Dampening, as mentioned, is almost always observed empirically due to the super- imposition of multiple continuity structures. Dampening is easily added by using the exponential structure as a multiplicative structure.
- The hole effect nested structure forces a zonal anisotropy in all other principle directions equal to the contribution of the hole effect. This may be a significant limitation, since empirical results are not limited to this very specific hole effect/zonal anisotropy configuration. Any attempt to work around this limitation by adding an additional structure, which is not present in the hole effect direction, leads to artifacts in the off diagonal directions. This is developed below.

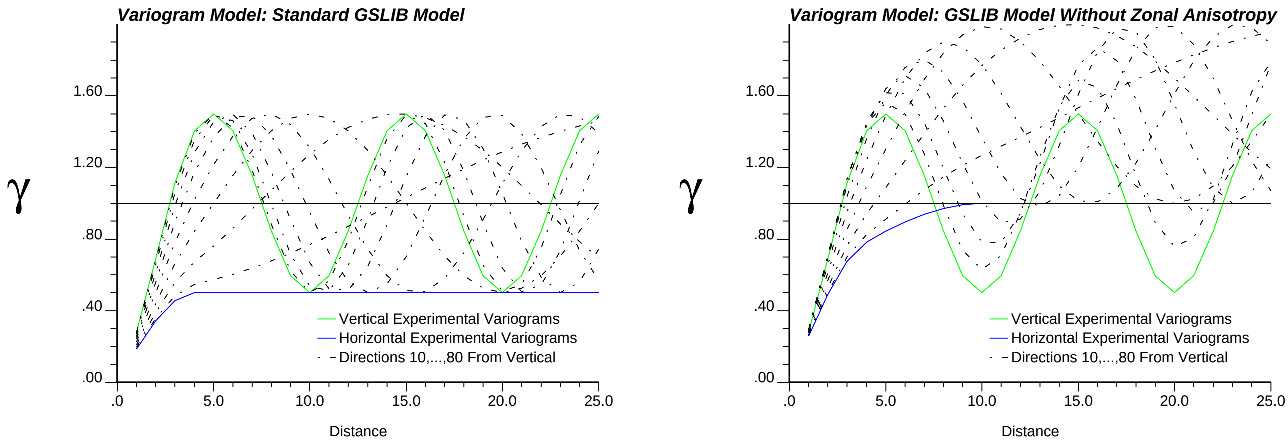

Figure 9: The GSLIB hole effect models with and without zonal anisotropy

The Inferred Continuity Structure in All Directions

A variogram model is required for all possible lag vectors, **h**. The GSLIB hole effect model with and without the previously mentioned minimum zonal anisotropy are displayed in Figure 9 with angles, $\phi = 0, 10, \ldots 90$ plotted. Where the variogram is modeled as:

0.2 Nugget + 0.6 Isotropic Spherical (Range = 10) + 0.2 Hole Effect (Range = 30)

For the GSLIB model without zonal anisotropy the addition of a structure, which acts in all directions except the hole effect direction is required. Some observations on the standard GSLIB model (1) the wavelength of the hole effect increases as the lag direction moves from parallel to perpendicular to the hole effect structure. The wavelength is scaled by a factor of $\frac{1}{cos(\phi)}$, and (2) the amplitude of the hole effect remains constant until the lag direction becomes perpendicular to the hole effect structure.

In the GSLIB model without zonal anisotropy the model amplitude increases as ϕ increases. This is caused by the additional contribution from the extra structure required to remove the zonal anisotropy.

A Synthetic Example

A comparison of the properties of the GSLIB hole effect variogram model to a ideal experimental hole effect variograms was conducted. A 132 x 132 grid and was constructed with a repeating string of 22 numbers. The numbers are normally distributed and start at -1.5 and reach 1.5, and then back to -1.5. This resulted in 6 repeats of the string in each row and a perfect hole effect. The extreme values were scaled to ensure the variance of the whole data set is equal to one (see Figure 10).

The GSLIB program, (GAMV), was used to calculate the variograms at the angles, $\phi = 0, 10, \ldots, 90$. The characteristics seen in the GSLIB hole effect model were reproduced with respect to constant amplitude and the wavelength scaled by $\frac{1}{cos(\phi)}$.

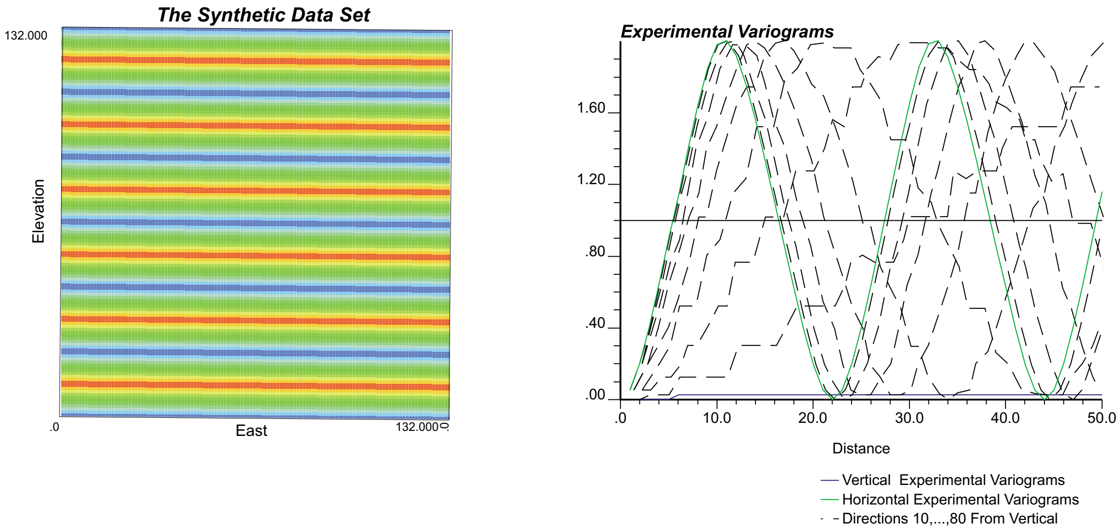

Figure 10: The synthetic data set and the resulting experimental variograms

It should be noted that the experimental variograms were calculated with a bandwidth less than the data spacing. This prevented the averaging of pairs that are not precisely along the lag vector. If the bandwidth is not restricted, the variograms in directions $10, \ldots, 80$ degrees from north express a reduced amplitude due to the impact of these additional off-lag pairs.

The hole effect model as implemented in GSLIB corresponds to a physically-plausible case. This is where the zonal anisotropy is equal to the hole effect contribution.

Kriging and Hole Effect Structures

Kriging is a best linear unbiased estimator. The resulting optimized point-wise estimates minimize estimation variance. Taken globally, kriged maps do not reproduce the one or two point statistics. The general form of the hole effect is reproduced in kriging although the variogram is distorted by the smoothing effect of kriging. These distortions include an increase in short scale continuity and dampening in the empiracal variograms.

Strong hole effect / zonal anisotropy continuity structures contradict the assumption of global stationarity inherent to the simple kriging (SK) algorithm. These structures are much more suited to the weaker dependence of local stationarity implicit to ordinary kriging (OK). Experimentally, it was found that these hole effect structures were better reproduced with the OK algorithm in estimation and simulation. For this reason all subsequent work utilized the OK algorithm. We note that this will cause some variance inflation in simulation due to the higher estimation variance of OK versus SK.

Simulation and Hole Effect Structures

Unlike kriging, simulation correctly reproduces the one and two point statistics in expected value. The reproduction of these statistics in the presence of a hole effect was investigated. Three examples of hole effect models and the resulting empirical variograms are shown in Figure 11.

The global mean and variance, in the presence of the hole effect, are reproduced in expected value.

Experimental variograms were calculated from three simulations and were compared with the original model variogram (refer to Figure 11). In expected terms the general form of the hole effect is reproduced. As the strength of the hole effect increases, so do the ergodic fluctuations in the experimental variogram.

The reproduction of the variogram is also affected by the search parameters. It is essential that a sufficient range and number of original data and previously simulated nodes are utilized to ensure that the auto-correlation between the simulated nodes reproduces the hole effect. Too small a search radius will "truncate" model structures in the experimental variograms.

The Sector Search

There are difficulties in the reproducing the variogram in the presence of hole effect. These difficulties result from estimates being made with too few data a range of covariances.

The current search methods focus on finding nearby conditioning data (original data and previously simulated nodes) with a selection criteria based on maximizing the level of autocorrelation between the conditioning data and the estimate location. This has the advantage of ensuring the minimal local uncertainty based on the available conditioning data and the variogram.

An alternative search method called a sector search routine has been proposed. In this context, sectors refer to the typical variogram search template. This technique would set a limit on the number of data sampled from each sector. This would result in the pooled data for simulating a specific location being representative of a variety of ranges and directions.

A sector search routine would not focus solely on minimizing the local uncertainty, but would focus on building a sample sets of conditioning data, which is representative of the population of autocorrelation relationships. This may result in an increase in local uncertainty, but will also result in better variogram reproduction.

Conclusions

The hole effect can be modeled and reproduced in simulation. In general, this is important when the variance contribution of the hole effect is more than 20%.

It is essential to validate that the one and two point statistics are adequately reproduced in expected terms. This may require some iteration of the search parameters.

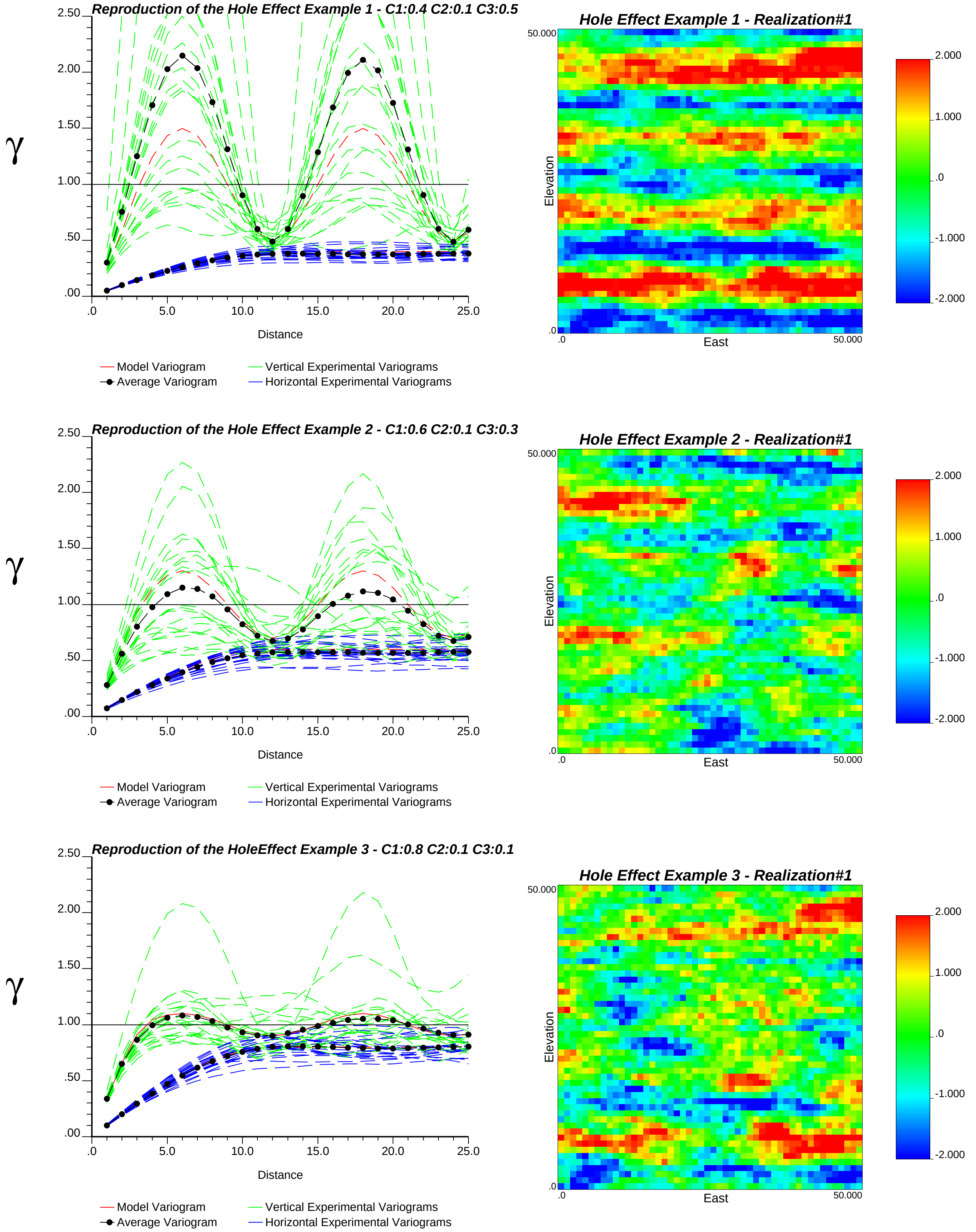

Figure 11: Three example variogram models, simulated variogram results, and one realization plot

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